BSc (Maths-I, paper-II

Equations of tangents and normal of general equation and their form in their particular conic section

~~Theorem~~ To find the equation of tangent to the conic

Equation of tangent

$f(x,y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ — (1)
at the point (x_1, y_1).

The equation of the tangent at point (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1) \quad \text{— (2)}$$

Now differentiating (1) w.r.t. x get

$$a2x + 2h\left(x\frac{dy}{dx} + y\right) + b \cdot 2y\frac{dy}{dx} + 2g \cdot 1 + 2f\frac{dy}{dx} + 0 = 0$$

$$\Rightarrow 2ax + 2hx\frac{dy}{dx} + 2hy + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$\Rightarrow 2\frac{dy}{dx}\{hx + by + f\} + 2\{ax + hy + g\} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}$$

Hence from (2) equation of the tangent at the point (x_1, y_1) is

$$y - y_1 = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}(x - x_1)$$

$\Rightarrow y(hx_1 + by_1 + f) - y_1(hx_1 + by_1 + \cancel{f})$

$= -x(ax_1 + hy_1 + g) + x_1(ax_1 + hy_1 + g)$

$\Rightarrow x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f)$

$= x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f)$

$= ax_1^2 + hx_1y_1 + gx_1 + hx_1y_1 + by_1^2 + fy_1$

$= ax_1^2 + 2hx_1y_1 + by_1^2 + gx_1 + fy_1$ — (3)

But (x_1, y_1) lies on the curve

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Therefore $ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0$

$\therefore ax_1^2 + 2hx_1y_1 + by_1^2 = -2gx_1 - 2fy_1 - c$

Hence the R.H.S of (3)

$= (-2gx_1 - 2fy_1 - c) + gx_1 + fy_1$

$= -gx_1 - fy_1 - c = -(gx_1 + fy_1 + c)$

Thus required equation of the tangent is

$x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f)$

$= -(gx_1 + fy_1 + c)$

ie $x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f)$

$+ gx + x_1 + fy + y_1 + c = 0$

which can also be arranged as

$axx_1 + byy_1 + h(xy_1 + x_1y) + g(x + x_1)$
$+ f(y + y_1) + c = 0$

Equation of the normal

To find the equation of the normal to the curve at the point (x_1, y_1)

Soln:- The equation of the curve is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ①

The slope of the tangent to the curve at the point (x_1, y_1) is $\frac{dy_1}{dx_1}$

The normal at the point (x_1, y_1) of the curve ① is defined as the straight line drawn through (x_1, y_1) perpendicular to the tangent at (x_1, y_1)

∴ From the formula $m_1 m_2 = -1$

The slope of normal to (x_1, y_1)

$= \frac{1}{\text{The slope of tangent at } (x_1, y_1)} = -\frac{dx_1}{dy_1}$

Hence the equation of the normal to ① at (x_1, y_1) is

$$y - y_1 = -\frac{dx_1}{dy_1}(x - x_1) \quad \text{②}$$

Differentiating ① with respect to x we get

$$2ax + 2hy + 2hx\frac{dy}{dx} + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

or $2\frac{dy}{dx}(hx + by + f) + 2(ax + hy + g) = 0$

or $\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}$

at the point (x_1, y_1),

$$\frac{dx_1}{dy_1} = -\frac{hx_1 + by_1 + f}{ax_1 + hy_1 + g}$$

From ② $y - y_1 = \frac{hx_1 + by_1 + f}{ax_1 + hy_1 + g}(x - x_1)$

or $\frac{x - x_1}{ax_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + by_1 + f}$

This is the required equation of the normal to the curve ① at the point (x_1, y_1)

particular cases of equation of tangent for conic sections in standard forms

Q:- To find the equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1)

In this case

$$f(x,y) \equiv y^2 - 4ax = 0 \quad -(1)$$

Hence diff. (1) w.r.to x we get

$$2y\frac{dy}{dx} - 4a = 0$$

$$\therefore \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

$$\frac{dy}{dx} \text{ at } (x_1, y_1) = \frac{2a}{y_1}$$

The equation of tangent at point (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

$\therefore$ The equation to the tangent at (x_1, y_1) on the parabola

$f(x,y) \equiv y^2 - 4ax$ is

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

or $-2a(x - x_1) + (y - y_1)y_1 = 0$

or $yy_1 - 2ax = -2ax_1 + y_1^2$ — (1)

As the point (x_1, y_1) lies on the parabola $y^2 = 4ax$

therefor $y_1^2 = 4ax_1$ — (2)

Hence from (1) and (2) the equation to the tangent at (x_1, y_1) is

$$yy_1 = 2ax - 2ax_1 + 4ax_1$$

or $$yy_1 = 2a(x + x_1)$$

<u>Corollary</u> The equation of the tangent to the parabola $y^2 = 4ax$ at the point t ie the point $(at^2, 2at)$

$$y\,2at = 2a(x + at^2)$$

or $$yt = x + at^2$$

Problem: The normal at the point $(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again in the point $(at_2^2, 2at_2)$ prove that

$$t_2 = -t_1 - \frac{2}{t_1}$$

Soln.:-

The eqn. to the tangent at the point 't_1' of the parabola $y^2 = 4ax$ is

$$yt_1 = x + at_1^2 \text{ —— } (1)$$

The gradient of (1) is $\frac{1}{t_1}$

If m be the gradient of the normal at 't_1' then

$m \times \frac{1}{t_1} = -1$ ∴ $m = -t_1$

The equation to the normal at the point $(at_1^2, 2at_1)$ is therefore $y - 2at_1 = -t_1(x - at_1^2)$

This line passes through the

point $(at_2^2, 2at_2)$

Hence $2at_2 - 2at_1 = -t_1(at_2^2 - at_1^2)$

or $2a(t_2 - t_1) = -at_1(t_2 - t_1)(t_2 + t_1)$

or $2 = -t_1(t_2 + t_1)$

or $\frac{2}{t_1} = t_2 + t_1$

or $t_2 = -t_1 - \frac{2}{t_1}$

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